Exercise 5

Find the general solution for the following second order ODEs:

$$u'' - 6u' + 9u = 0$$

Solution

This is a homogeneous linear ODE with constant coefficients, so the solution is of the form, $u = e^{rx}$.

$$u = e^{rx} \quad \rightarrow \quad u' = re^{rx} \quad \rightarrow \quad u'' = r^2 e^{rx}$$

Substituting these into the equation gives us

$$r^2 e^{rx} - 6re^{rx} + 9e^{rx} = 0.$$

Divide both sides by e^{rx}.

$$r^2 - 6r + 9 = 0$$

Factor the left side.

$$(r-3)^2 = 0$$

$r = 3$ with a multiplicity of 2. Therefore, the general solution is

$$u(x) = C_1 e^{3x} + C_2 x e^{3x}.$$

We can check that this is the solution. The first and second derivatives are

$$\begin{aligned} u' &= e^{3x}(3C_1 + C_2 + 3C_2 x) \\ u'' &= 3e^{3x}[3C_1 + C_2(2 + 3x)]. \end{aligned}$$

Hence,

$$u'' - 6u' + 9u = 3e^{3x}[\cancel{3C_1} + C_2(\cancel{2} + \cancel{3x})] - 6e^{3x}(\cancel{3C_1} + \cancel{C_2} + \cancel{3C_2 x}) + 9e^{3x}(\cancel{C_1} + \cancel{C_2 x}) = 0,$$

which means this is the correct solution.